Determining Q Using S Parameter Data

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***Abstract*—A method is presented for determining frequency selectivity (Q) of a network using scattering (S) parameter data, data that is readily available from network measurements or analysis. The approach is based on a formulation for Q that uses the change in reactance of the resonant circuit with frequency. The method yields accurate Q results for both high and low Q resonators. Furthermore, the method is easy to implement and to understand. An example is given for calculating the Q of a tapped-stub resonator. Using this example, the new method is compared to the critical points (CP) method, an approach based on a Foster network type of formulation.**

I. Introduction

Determining the Q of a resonator is very important for describing the frequency-selective performance of a filter. Accordingly, it is desirable to have a technique for determining Q that is both easy to implement and accurate. Furthermore, since network analyzers and transmission line modeling software are prevalent, this technique should be able to use data readily available from these sources. In this article we present a method for determining Q which meets these needs.

The method presented is derived directly from the fundamental definition of Q without any approximations. Because the method derives directly from the fundamental definition, the method does not impose any structure on the resonator being measured, such as a simple R–L–C circuit representation. Thus, the method is able to accurately determine the Q for transmission line circuits and other resonators which have multiple pole structures. As a result, it is accurate for both high Q and low Q circuits.

The only information required by the method is S parameter data at frequencies near the resonant point of the resonator. Since S parameter data is readily available from network analyzer measurements or from network analysis or modeling software, the method is easy to implement. For this article we discuss and use only S_{11} data to determine Q, but there is an equally valid formulation using S_{21}. The present formulation can easily be extended to use other equivalent forms of S parameter data (two-port network parameters), such as ABCD matrix data and Y, Z parameter data.

There are a number of other methods for determining Q [1] and [2]. A few recent papers have presented methods that use the network analyzer to measure Q [3] and [4]. However, these methods have several limitations. One limitation is that several of these methods require reading information from a Smith Chart which is both cumbersome and subject to user error. Another limitation of these methods is that they use approximations in their formulations that create errors in low Q resonators. A final limitation to other methods is that they are less tractable to a novice user, and thus less likely to be implemented based on their complexity.

In this article we begin by providing a formulation for Q that is the basis for the method presented. We then derive the equations needed to implement this method using S_{11} data. This is followed by an example of a tapped-stub resonant circuit which serves to illustrate the implementation of the method. We conclude by comparing the results obtained from this method with the critical points (CP) method [4] on three different tapped-stub resonator filters.

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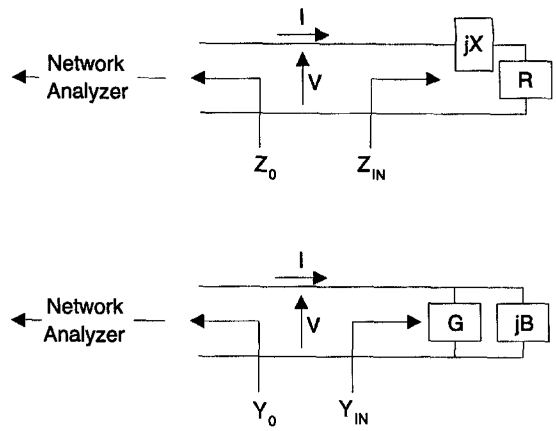

Fig. 1. General circuit representations.

II. Definition of Q

We begin our discussion by stating the fundamental definition of Q

$$Q = \omega_0 \frac{\text{Peak Energy Stored}}{\text{Average Power Lost}}. \tag{1}$$

In (1), $f_0 = \omega_0/(2\pi)$ is the resonant frequency, or the frequency at which Q is evaluated. The numerator is the peak energy stored in either the electric or the magnetic field, and the denominator is the average power dissipated within or coupled out of the network. The problem with this fundamental definition, as with many fundamental definitions, is that (1) does not lend itself to making physical measurements. Thus, we need to create a representation of Q in terms of measurable parameters that is based on this fundamental definition.

A. Representation of Q

The representation of Q to be used as the basis for the present method begins with representing a general circuit in terms of a series impedance Z_{in} expressed as,

$$Z_{in} = R + jX \tag{2}$$

or a parallel admittance Y_{in} expressed as

$$Y_{in} = G + jB \tag{3}$$

where R, X, G, and B are all functions of frequency, ω. These are shown pictorially in Fig. 1. In Fig. 1, we represent the source resistance by Z_0 and the source conductance by Y_0. In doing so, we implicitly assume that the source resistance or conductance is real and does not vary with frequency. Altering this assumption does not change our approach, but it does increase the complexity of the equations used to determine Q.

The reactance X may also be expressed as [5]

$$X = \frac{4(W_E - W_H)}{II^*} \tag{4}$$

and the susceptance B as

$$B = \frac{4(W_E - W_H)}{VV^*} \tag{5}$$

where W_E and W_H are the stored electric and magnetic energies, respectively, in the system; and I and V are the input current and voltage, respectively.

Furthermore, it can be shown [5] that

$$\frac{\partial X}{\partial \omega} = 4\,\frac{W_E + W_H}{II^*} \tag{6}$$

and

$$\frac{\partial B}{\partial \omega} = 4\,\frac{W_E + W_H}{VV^*} \tag{7}$$

where $W_E + W_H$ is the total energy stored in the system. This total energy stored in the system reaches a maximum at the resonant frequency, $\omega = \omega_0$. Thus

$$\begin{aligned}\text{Peak Energy Stored} &= \left[\frac{II^*}{4}\,\frac{\partial X}{\partial \omega}\right]_{\omega=\omega_0} \\ &= \left[\frac{VV^*}{4}\,\frac{\partial B}{\partial \omega}\right]_{\omega=\omega_0}.\end{aligned} \tag{8}$$

The average power lost by the system is given by

$$\begin{aligned}\text{Average Power Lost} &= \tfrac{1}{2}\,II^*(R + Z_0) \\ &= \tfrac{1}{2}\,VV^*(G + Y_0)\end{aligned} \tag{9}$$

where the total series resistance is $R + Z_0$, and the total parallel conductance is $G + Y_0$.

Substituting (6) and (9) into the fundamental definition of Q in (1), we obtain the following representations for the loaded Q

$$Q = \left[\frac{\omega}{2(R + Z_0)}\,\frac{\partial X}{\partial \omega}\right]_{\omega=\omega_0} \tag{10}$$

and

$$Q = \left[\frac{\omega}{2(G + Y_0)}\,\frac{\partial B}{\partial \omega}\right]_{\omega=\omega_0}. \tag{11}$$

The technique developed herein for determining Q is based on the expressions given in (10) and (11). These expressions are also given in [6, p. 414].

B. Other Q Determination Techniques

There are a number of other methods that may be used to determine Q. Only two will be mentioned here for comparison purposes. A simple and often used expression for determining Q is

$$Q = \frac{f_0}{f_2 - f_1} \tag{12}$$

where f_0 is the resonant frequency, and f_2 and f_1 are the upper-frequency and the lower-frequency 3-dB points, respectively. This expression is derived by assuming that the resonator has a simple R–L–C or G–L–C circuit equivalent. This expression is valid for resonators with a single resonant point. However, for resonators with multiple resonant points this expression is approximate because other resonances will affect the 3-dB values of the particular resonance being considered. Thus, for multiple resonant point structures such as transmission lines the method expressed in (12) is only valid for high Q circuits where resonant points are well isolated. For low Q circuits this expression is very inaccurate. This will be shown to be the case by an example in Section V.

Other Q determination techniques have relied on a Foster network type of formulation. The equivalent circuit for a Foster network can be expressed as [4]

$$Z_{in} = R_e + j\omega L_e + \frac{R_0}{1 + jQ_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)} \tag{13}$$

where R_e and L_e are the external elements to the resonant points being measured. The goal in using this formulation is to extract Q_0. This is typically done using information from the Smith Chart. However, reading information from a Smith Chart can lead to inaccurate results. In addition, methods for determining Q that are based on this formulation use the approximation

$$\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \cong 2\,\frac{\omega - \omega_0}{\omega_0}. \tag{14}$$

This approximation causes these techniques to lose accuracy for low Q circuits. This fact will be illustrated in Section V where the present method (developed herein) is compared to the CP method, a method that uses the Foster network formulation in (13).

III. Procedure for Determining Q Using S_{11}

As mentioned in Section II-A, the basis for the method presented here is the representation of Q given in (10) and (11). With these expressions, Q can be determined by evaluating $\partial X/\partial \omega$ and R or, in terms of admittance, $\partial B/\partial \omega$ and G at the resonant frequency, ω_0. To evaluate these terms at the resonant frequency we use S_{11} data.

We begin by converting the S_{11} data into the form of (2) or (3). Starting with the definition of S_{11}

$$\begin{aligned}S_{11} &= \frac{Z_{in} - Z_0}{Z_{in} + Z_0} \\ &= \frac{Y_0 - Y_{in}}{Y_0 + Y_{in}}.\end{aligned} \tag{15}$$

Solving for Z_{in} and Y_{in} in terms of S_{11}

$$Z_{in} = Z_0\,\frac{1 + S_{11}}{1 - S_{11}} \tag{16}$$

and

$$Y_{in} = Y_0\,\frac{1 - S_{11}}{1 + S_{11}}. \tag{17}$$

S_{11} has both real and imaginary terms represented as

$$S_{11} = \mathrm{Re}\,[S_{11}] + j\,\mathrm{Im}\,[S_{11}]. \tag{18}$$

Substituting (18) into (16) and (17) and expanding both expressions into real and imaginary terms yields

$$\begin{aligned}Z_{in} = Z_0\,&\frac{\{(1 + \mathrm{Re}\,[S_{11}])(1 - \mathrm{Re}\,[S_{11}]) - \mathrm{Im}\,[S_{11}]^2\}}{(1 - \mathrm{Re}\,[S_{11}])^2 + \mathrm{Im}\,[S_{11}]^2} \\ &+ jZ_0\,\frac{2\,\mathrm{Im}\,[S_{11}]}{(1 - \mathrm{Re}\,[S_{11}])^2 + \mathrm{Im}\,[S_{11}]^2}\end{aligned} \tag{19}$$

and

$$\begin{aligned}Y_{in} = Y_0\,&\frac{\{(1 + \mathrm{Re}\,[S_{11}])(1 - \mathrm{Re}\,[S_{11}]) - \mathrm{Im}\,[S_{11}]^2)\}}{(1 + \mathrm{Re}\,[S_{11}])^2 + \mathrm{Im}\,[S_{11}]^2} \\ &+ jY_0\,\frac{-2\,\mathrm{Im}\,[S_{11}]}{(1 + \mathrm{Re}\,[S_{11}])^2 + \mathrm{Im}\,[S_{11}]^2}.\end{aligned} \tag{20}$$

By comparing terms of (19) with (2)

$$R = Z_0\,\frac{\{(1 + \mathrm{Re}\,[S_{11}])(1 - \mathrm{Re}[S_{11}]) - \mathrm{Im}\,[S_{11}]^2\}}{(1 - \mathrm{Re}\,[S_{11}])^2 + \mathrm{Im}\,[S_{11}]^2} \tag{21}$$

$$X = Z_0\,\frac{2\,\mathrm{Im}\,[S_{11}]}{(1 - \mathrm{Re}\,[S_{11}])^2 + \mathrm{Im}\,[S_{11}]^2} \tag{22}$$

or comparing (20) with (3)

$$G = Y_0\,\frac{\{(1 + \mathrm{Re}\,[S_{11}])(1 - \mathrm{Re}[S_{11}]) - \mathrm{Im}\,[S_{11}]^2)\}}{(1 + \mathrm{Re}\,[S_{11}])^2 + \mathrm{Im}\,[S_{11}]^2} \tag{23}$$

$$B = Y_0\,\frac{-2\,\mathrm{Im}\,[S_{11}]}{(1 + \mathrm{Re}\,[S_{11}])^2 + \mathrm{Im}\,[S_{11}]^2}. \tag{24}$$

Equations (21) and (23) allow us to calculate R and G, respectively, at the resonant frequency directly from S_{11} data. The next step is to

TABLE I
STEPS IN DETERMINING Q USING S_{11} DATA

Step	Z_{in} Network Representation	Y_{in} Network Representation
1	Collect S_{11} data	Collect S_{11} data
2	Convert S_{11} data into Re[S_{11}] and Im[S_{11}] form, if needed	Convert S_{11} data into Re[S_{11}] and Im[S_{11}] form, if needed
3	Find resonant frequency, f_0	Find resonant frequency, f_0
4	Solve for R using (21) at the resonant frequency	Solve for G using (23) at the resonant frequency
5	Solve for X using (22) around the resonant frequency	Solve for B using (24) around the resonant frequency
6	Use (25) to solve for $\frac{\partial X}{\partial \omega}$	Use (26) to solve for $\frac{\partial B}{\partial \omega}$
7	Use (10) to determine Q	Use (11) to determine Q

calculate $\partial X/\partial\omega$ and $\partial B/\partial\omega$ at the resonance frequency, ω_0, using (22) and (24). Since S_{11} is a function of frequency, ω, (22) and (24) are functions of frequency, and thus we use the functional form, $S_{11}(\omega)$, to find $X(\omega)$ and $B(\omega)$ and then differentiate with respect to ω. However, S_{11} data is typically given in discrete form, such as that provided by a network analyzer. Thus, a numerical derivative is required. This numerical derivative is typically straightforward since $X(\omega)$ and $B(\omega)$ are fairly linear functions of frequency at the resonance point. As a result the slope can be calculated with a simple regression formula. The following equations are used to calculate the slope at the resonant point from discrete $X[\omega]$ and $B[\omega]$ data

$$\frac{\partial X}{\partial \omega} \cong \frac{n(\Sigma\,\omega X) - (\Sigma\,\omega)(\Sigma X)}{n(\Sigma\omega^2) - (\Sigma\omega)^2} \tag{25}$$

and

$$\frac{\partial B}{\partial \omega} \cong \frac{n(\Sigma\omega B) - (\Sigma\omega)(\Sigma B)}{n(\Sigma\omega^2) - (\Sigma\omega)^2} \tag{26}$$

where n is the number of discrete points of data used to calculate the slope.

The final step is to substitute the information from (25) and (21) at the resonant point into (10) to yield Q. Likewise, using admittances, (26) and (23) at the resonant frequency are substituted into (11) to yield Q. Table I summarizes the steps used to calculate Q from S_{11} data. An example follows.

IV. TAPPED-STUB RESONATOR EXAMPLE

The following is an example for determining the Q for a microstrip bandpass filter. For this example we chose a tapped-stub resonator circuit [6]. A tapped-stub resonator is created by changing the connection point of a stub on the main transmission line. This is illustrated in Fig. 2.

The particular two-port, tapped-stub resonator shown in Fig. 2 is referred to as a half wavelength tapped-stub resonator because the total length of the stub is $l = \lambda_0/2$ at resonance. Thus, the resonant condition is given by

$$\theta_1 + \theta_2 = \pi\,\frac{\omega}{\omega_0}. \tag{27}$$

We introduce a tapping factor, k, which gives the proportional length of the stub on each side of the main line where $0 \le k < 1$ such that

$$\theta_1 = k\,\frac{\pi\omega}{2\omega_0} \tag{28}$$

and from (27)

$$\theta_2 = (2-k)\,\frac{\pi\omega}{2\omega_0}. \tag{29}$$

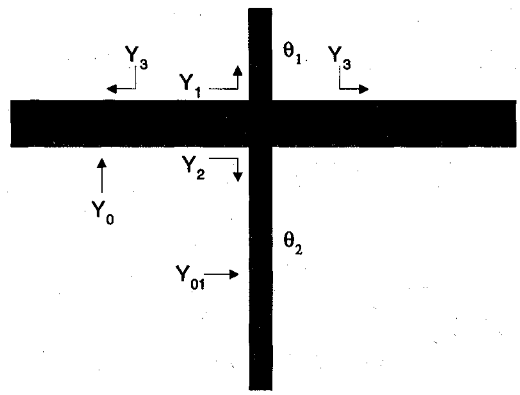

Fig. 2. Two-port, half-wavelength tapped stub resonator.

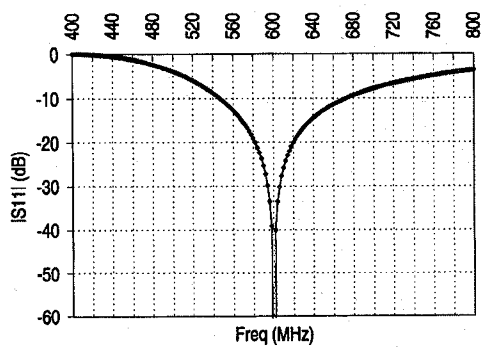

Fig. 3. S_{11} magnitude for tapped-stub resonator with $k = \frac{1}{2}$.

Note that $k = 0$ represents an untapped half wavelength open stub, and $k = 1$ represents a pair of quarter wavelength open stubs. The tapped-stub is well suited as an example because as the tapping factor k increases, the Q of the filter increases without significantly affecting the resonant frequency of the filter. The Q of this circuit is calculated as follows.

The node admittance of the two-port half-wavelength tapped stub is given by

$$Y = 2Y_3 + Y_1 + Y_2. \tag{30}$$

Substituting the admittance values

$$Y = 2Y_0 + jY_{01}(\tan\theta_2 + \tan\theta_1). \tag{31}$$

TABLE II
CALCULATION OF $\partial B/\partial f$ FOR $k = \frac{1}{2}$

f (MHz)	|S11|	∠S11	Re[S11]	Im[S11]	G	B
590	5.40E-02	93.1	-0.00292	0.053921	0.02	-0.00216
592	4.30E-02	92.5	-0.00188	0.042959	0.020001	-0.00172
594	3.20E-02	91.8	-0.00101	0.031984	0.019999	-0.00128
596	2.10E-02	91.2	-0.00044	0.020995	0.02	-0.00084
598	1.10E-02	90.6	-0.00012	0.010999	0.02	-0.00044
600	5.90E-10	90.2	-2.1E-12	5.9E-10	0.02	-2.4E-11
602	1.00E-02	-90.6	-0.0001	-0.01	0.02	0.0004
604	2.10E-02	-91.2	-0.00044	-0.021	0.02	0.00084
606	3.10E-02	-91.8	-0.00097	-0.03098	0.020001	0.001241
608	4.10E-02	-92.4	-0.00172	-0.04096	0.020001	0.001641
610	5.10E-02	-92.9	-0.00258	-0.05093	0.019999	0.002043
					$\partial B/\partial f$	2.10×10^{-10}

Thus, the conductance is given by

$$G = 2Y_0. \quad (32)$$

Using (28) and (29), the susceptance is given by

$$B = Y_{01}\left[\tan k \frac{\pi\omega}{2\omega_0} + \tan(2-k)\frac{\pi\omega}{2\omega_0}\right]. \quad (33)$$

From (33), the partial derivative of B with respect to ω evaluated at ω_0 is

$$\left.\frac{\partial B}{\partial \omega}\right|_{\omega=\omega_0} = \frac{\pi Y_{01}}{2\omega_0}\left[k \sec^2 k\frac{\pi}{2} + (2-k)\sec^2(2-k)\frac{\pi}{2}\right]. \quad (34)$$

Using (11), (32), and (34), the Q of the two-port, half wavelength tapped-stub resonator is given by

$$Q = \frac{\pi Y_{01}}{4Y_0}\sec^2 k\frac{\pi}{2}. \quad (35)$$

Examination of (35) reveals that as $k \to 0$, $Q \to \pi Y_{01}/4Y_0$ and as $k \to 1$, $Q \to \infty$. The equivalent expression for a one-port, half wavelength tapped-stub resonator is given in [6, p. 426].

As an illustrative case we have chosen a circuit with $k = \frac{1}{2}$, $Y_0 = 50\,\Omega$, $Y_{01} = 50\,\Omega$, and $f_0 = 600$ MHz. The value of $k = \frac{1}{2}$ was chosen for two primary reasons. First, the theoretical Q is low which will allows us to validate the new method for low Q circuits. Second, this circuit can be easily implemented on microstrip with minimal losses.

The first step is to collect S_{11} data around the resonant frequency. The data we collected for this example is from a transmission line software package called "Puff." Fig. 3 shows S_{11} magnitude data for this example and Fig. 4 shows S_{11} phase data around the resonant frequency, $f_0 = 600$ MHz.

The next step is to use (21) and (22), or (23) and (24), depending on the desired circuit representation. As indicated earlier, for the tapped-stub resonator, representing the circuit in terms of an input admittance Y_{in} is appropriate, and thus we use (23) and (24). Table II shows the S_{11} data that was collected around the resonant frequency. This data is in terms of a magnitude and phase angle. We also show in this table the S_{11} data transformed into its real and imaginary components, and finally using (23) and (24), we show the S_{11} data converted into G and B.

Next using (26), the derivative of B with respect to frequency is approximated. Fig. 5 shows B versus $f = \omega/2\pi$ around the resonant frequency f_0 for the modeled results. The derivative, $\partial B/\partial\omega$ is calculated by finding the slope of this line and dividing by 2π. To calculate the slope, 11 points are used, or $n = 11$. This yields a slope $\partial B/\partial f \cong 2.10 \times 10^{-10}$, which corresponds to $\partial B/\partial\omega \cong 3.34 \times 10^{-11}$. The solid line in Fig. 5 represents this slope approximation for the modeled data. We have also included in Fig. 5 data from an actual circuit measured on a network analyzer. Finally, (11) is used to evaluate Q. The value of G at the resonant frequency is used for this calculation, as $G = 0.02\,S$. For our circuit the modeled results yield $Q = 1.576\,559$.

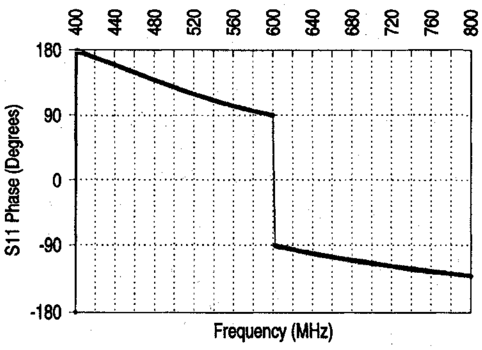

Fig. 4. S_{11} phase for tapped-stub resonator with $k = \frac{1}{2}$.

TABLE III
COMPARISON BETWEEN Q MEASUREMENTS FOR $k = \frac{1}{2}$

	Q
Theory	1.570796
Modeled	1.576559
Measured	1.480502
Modeled with Loss	1.476844
3-dB Points Method	0.566667

Table III compares the Q determined from theory, from the transmission line modeling software, and from network analyzer measurements on an actual circuit.[1] Notice that the measured results are significantly below the theoretical and modeled results. This is due to losses in the circuit that have not been accounted for in the model. Losses will lower the Q of a circuit because losses increase the *Average Power Lost* in the denominator of (1). As a result, we have treated the case where losses were added to the transmission line

[1]For the measured circuit Rogers RT Duroid 5880 with two-sided 1-oz rolled copper cladding was used. The relative dielectric constant was 2.2 and the substrate thickness was 1/8 in. Network analyzer measurements were made with a Hewlett Packard 8753A.

TABLE IV
COMPARISON OF Q USING DIFFERENT METHODS

Case	Theory	Current Method	Critical Points	3-dB Points
$k = 1/2$	1.570796	1.576559	0.958023	0.566667
$k = 9/10$	32.09409	32.13687	28.2282	15.38462
$k = 99/100$	3183.36	3189.031	3187.96	3191.489

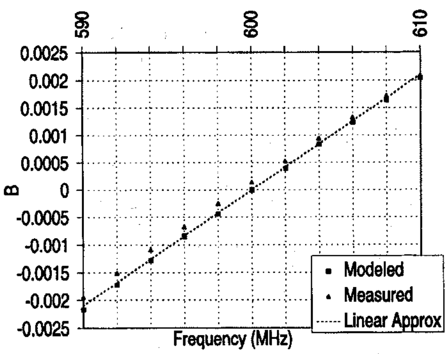

Fig. 5. B versus frequency for modeled and measured, $k = \frac{1}{2}$.

modeling software. A loss tangent of 0.05 is used because this level of loss is consistent with the losses associated with a microstrip circuit and the network analyzer losses. Note that the Q for the modeled results with these losses is in fairly good agreement with the measured results. For comparison we have also shown the results using the 3-dB Points method. As expected, this yielded an inaccurate result for this low-Q circuit.

V. Comparison to Critical Points Method

In this section we compare the current method with both the CP method and the 3-dB points method. For this comparison we chose three cases of the tapped-stub resonant circuit, $k = \frac{1}{2}$, $k = \frac{9}{10}$, and $k = \frac{99}{100}$. By (35), increasing k will increase Q. Table IV shows the results from theory, the current method, the CP method, and the 3-dB Points method.

Clearly this comparison indicates that the approach developed herein using S-parameter data is valid for a wide range of Q values. The CP method is valid at higher values of Q but loses accuracy at low values of Q. The inaccuracy at low values of Q is largely due to the approximation given in (14). It can also be attributed to errors in reading the Smith chart. The 3-dB Points method is clearly inferior to both the current method and the CP method. The 3-dB Points method can only be used reliably for high-Q resonators with well isolated resonant points and for resonators with a single resonant point.

VI. Conclusion

In this article we have developed a technique for determining Q that is both accurate and easy to implement. It has proven to be accurate for both low-Q and high-Q resonators. In fact we have demonstrated that it is more accurate than another method, the CP method, that relies on a Foster type of circuit formulation. The formulation for this method is very straightforward which makes this method tractable for a novice user. Furthermore, because the method uses S-parameter data that is easy to obtain, this method can be implemented with relatively little effort.

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